**Quantum Mechanics of Fields**

So quantum mechanics considers the fundamental constituents of the universe to be an assortment of particles. And these particles interact with each other via fields that they sort of just ‘create’. Quantum Field Theory flips this picture around. It considers the field to be fundamental, and particles to be simply excitations of the field. This makes it easier to conceive, for instance, how two particles interact. They don’t create the field; the field creates them.

**Postulate 1: Hilbert Space**

A particle exists in a Hilbert space |x>, where x can be any real #, and so there are a continuously infinite # of basis vectors. Multiple particles exist in a HS = |x1>|x2>…|xN>, which would be an N-fold product of an infinite # of bases. It’s probably best to think of it as an extension of the HS of a many-body harmonic oscillator – where there is an independent basis |**r**> at every point, **R**, where an oscillator is located. So note there is a basis vector for every configuration of the system. A possible state for a quantum field is any set of values at every point. And so the basis would look like this I think,



where |φ> is a basis vector corresponding to a given field configuration φ(**x**), say. If we’re dealing with a fermionic field, ψ(**x**), then we have to include spinor degrees of freedom too, since the field has 4 components. So we’d say:



The overlap between two such basis kets would be, presumably,



i.e., we get non-zero overlap only if the two fields are identical at every point. A general field vector would be represented by the state ket |Ψ> (not to be confused with fermion field). And the wavefunction would be a functional, something which associates a probability amplitude to each field configuration. So it could look like, e.g.



where again φ(x) would be a possible configuration. For instance, we might have something like:



where v(x,y) and u(x) are known functions of some sort. And we may suppose that:



Maximizing |<φ|Ψ>|2 = |Ψ[φ(**x**)]|2 – i.e. taking functional derivative w/r to φ(**x**) and setting to zero, etc. – would give us the most likely field configuration, just like how in regular quantum mechanics maximizing |<**x**|Ψ>|2 = |Ψ(**x**)|2 would give us the most likely position. As befits a probability density, we must have that:



Just like particles in a given potential experience certain excitations, so do quantum fields. And it is usually convenient to consider a Fock-space like representation of the HS in terms of the field’s excitations. In many cases the analogy is pretty much exact, and we can write the fields in terms of field creation/annihilation operators.

**Postulate 2: Representation of observables**

Operators are mappings from one state to another. So I guess we could say, in the position basis:



Just making sure that multiplying two resolutions of identity gives me back a resolution of identity…



So that checks out. Let’s consider an elastic medium for example. I’d like to know how one represents the field configuration, momentum, kinetic energy, and elastic potential energy. So first let’s designate as an operator which pulls out the field value at the specific point x. So that:



and so that means we can represent the field configuration operator, in the field configuration basis as:



So we have:



We can easily see that these operators commute.



As a consequence these can be simultaneously diagonalized. And we can say that the state vector |φ>, as we’ve been discussing it, is the state ket which simultaneously diagaonalizes the I’ll insert a discussion about the FT of this operator, since we’ll be invoking it later on. Let’s define:



This seems not to be well-defined: how do you ‘add up operators’ and expect to get something in the continuum limit? Maybe it’s the d3x factor which helps convergence. Anyway, we’ll just treat it as the analogue to a discrete FT of HO position operators, like = ΣRe-ik·R. In the discrete case there would be a unique set of N such k-operators, just as there is a unique set of N such position operators. Now since the can be simultaneously diagonalized, so can the . What are their eigenvalues/states? Eigenstates seem to simply be |φ> itself/themselves, since:



And so the eigenvalues are obviously ∫d3x e-i**k**·**x**φ(**x**). The operator canonically conjugate to , i.e. , would satisfy the canonical commutation relation:



[setting h-bar to 1] What does this operator look like? I think by analogy with QM we may assume,



This could probably be demonstrated by introducing a ‘translation’ operator – kind of a misnomer in this context – but we can also just directly test the commutation relation.



So there. We can have operators for the energy of a continuous substance, like a set of harmonic oscillators in the continuum limit. In that case φ(**x**) would represent the displacement of the oscillator at **x**, and π(**x**) would represent its momentum (density). Then the Hamiltonian would be look like, using a homogeneous elastic solid for instance:



(square really means the magnitude of that vector) Eigenfunctions of H would give the allowed energies of the field. We’d have to solve:



Projecting the action of these operators on a generic wavevector against a field configuration basis vector would give us:



and,



Can verify that doing the same with our generic H above would work out to:



Now recall <φ|Ψ> is a functional. And I guess this would be a functional differential equation. I can’t say I know how to solve that. And this is probably one reason to abandon the wavefunction-centric approach to quantum mechanics when dealing with fields.

**Example**

What are the simultaneous eigenfunctions, |π>, of the momentum density operator(s) in the field configuration basis? Well let’s start off with the eigenvalue equation:



Projecting both sides against <φ| we have:



It’s not hard to see that the solution is:



Not sure about the normalization…

**Example**

Let’s say we have a wavefunction in the field basis:



What does it look like in the momentum basis?



This integral can actually be done. We have:



where v-1(**x**,**y**) is defined via:



and the determinant thing can be written as:



**Postulate 3: Time development**

In the Schrodinger picture, we’d have the usual:



And we can introduce the time development operator,



which would obey the differential equation:



and have an iterative solution we can formally resum to get:



which can be profitably written, in the interaction picture, as:



(for alternative phase conventions see QM folder) For operators, we define time-development according to the usual,



Taking the derivative w/r to time, we find we can write the time-development as a differential equation, just like before,



or more conveniently,



while in the Heisenberg picture, we can also just appropriate the classical equations of motion, as before, which will follow from the usual derivatives on the Action, etc. Along those lines, more often than not, we start off knowing the action of the fields, because we know what it is classically. We don’t typically start from the action in regular quantumm mechanics because we don’t typically use the Heisenberg picture, and because it isn’t easiest to get excitations and such from the Heisenberg picture in that context. But it is easier here. So anyway, let’s say we have some action S which depends on our field operator (**x**,t) (implicitly in the Heisenberg picture, but leaving the subscript off), and its derivatives ∂μ(**x**,t), where ∂μ = (∂t, ∇). Let’s consider a generic scalar field action.



The equation of motion would follow from the least action principle just like usual. We would take the functional derivative and set it to 0…so we’d have (implicit summation over μ):



and so we have (implicit summation over μ):



With the action/Lagrangian one can can construct H if it were not a priori known. This would be:



and one can verify that the equations of motion that come from minimizing the action are the same as come from the commutation relation.



and that are satisfied by the formal solution:



as was remarked upon in the usual quantum mechanics folder.